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The laminar wall-jet over a curved surface

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The laminar flow of a wall jet over a curved surface is considered. A unique similarity solution is obtained for both concave and convex surfaces when the local radius of curvature is proportional to $x^{\frac{3}{4}}$. This solution satisfies a similar invariant condition to the one derived by Glauert for the wall jet over a plane surface. The variation of the shape of the velocity profile, the skin friction, and the surface pressure as a function of curvature is given.

Introduction

Jets are observed to adhere to and follow the curvature of a solid surface. This phenomenon, which is accompanied by a pressure difference across the jet, is often referred to in the literature as the Coanda effect. Since this effect has a variety of applications, considerable attention was given to its investigation, both in theory and experiment, but only turbulent flow was considered. Newman (1961), Nakaguchi (1961) and Fekete (1963) investigated the flow of a jet around a circular cylinder, while Sawyer (1962) and Guitton (1964) examined the flow around a logarithmic spiral as well. Because of its complexity, the flow is hardly amenable to theoretical analysis and a large number of assumptions had to be introduced to predict some gross properties of the phenomenon. Newman (1961) used a momentum integral technique, neglected skin friction, and replaced the actual velocity profile with a rectangular one having the same mass and momentum flux. Nakaguchi (1961) assumed the velocity profiles similar and represented by the same function as the free jet in ambient fluid. Sawyer (1962) and Guitton (1964) determined the effect of wall curvature on the velocity but they assumed the wall to be frictionless. They found that the equations governing the flow of a turbulent jet over a logarithmic spiral may be reduced to self-preserving form after a suitable assumption relating the shear stress to the mean velocity is made. Their analysis, however, is of the perturbation type and only the first-order term was obtained. Even then, two empirical constants are required to determine the velocity profile.

The purpose of the present paper is to examine the conditions for which the flow of a jet over a curved surface is amenable to analysis with the smallest number of *ad hoc* assumptions and to obtain a solution satisfying these conditions. The flow considered is laminar, incompressible and two-dimensional. The jet is immersed in an identical ambient fluid which is at rest.

Analysis

The equations of motion and continuity for two-dimensional incompressible, laminar flow along a curved wall expressed in curvilinear orthogonal co-ordinates are (Goldstein 1938):

$$\left.\begin{aligned}\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\left(1+\frac{\tilde{y}}{\tilde{R}}\right)\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}+\frac{\tilde{u}\tilde{v}}{\tilde{R}} &= -\frac{1}{\rho}\frac{\partial p}{\partial\tilde{x}}+\nu\left\{\frac{1}{\left(1+\frac{\tilde{y}}{\tilde{R}}\right)}\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}+\left(1+\frac{\tilde{y}}{\tilde{R}}\right)\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}+\frac{1}{\tilde{R}}\frac{\partial\tilde{u}}{\partial\tilde{y}}\right.\\ &\left.-\frac{\tilde{u}}{\tilde{R}^2\left(1+\frac{\tilde{y}}{\tilde{R}}\right)}+\frac{2}{\tilde{R}\left(1+\frac{\tilde{y}}{\tilde{R}}\right)}\frac{\partial\tilde{v}}{\partial\tilde{x}}-\frac{1}{\tilde{R}^2\left(1+\frac{\tilde{y}}{\tilde{R}}\right)^2}\frac{\partial\tilde{R}}{\partial\tilde{x}}\tilde{v}+\frac{\tilde{y}}{\tilde{R}^2\left(1+\frac{\tilde{y}}{\tilde{R}}\right)^2}\frac{d\tilde{R}}{d\tilde{x}}\frac{\partial\tilde{u}}{\partial\tilde{x}}\right\},\end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned}\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{x}}+\left(1+\frac{\tilde{y}}{\tilde{R}}\right)\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{y}}-\frac{\tilde{u}^2}{\tilde{R}} &= -\left(1+\frac{\tilde{y}}{\tilde{R}}\right)\frac{1}{\rho}\frac{\partial p}{\partial\tilde{y}}+\nu\left\{\left(1+\frac{\tilde{y}}{\tilde{R}}\right)\frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}-\frac{2}{\tilde{R}\left(1+\frac{\tilde{y}}{\tilde{R}}\right)}\frac{\partial\tilde{u}}{\partial\tilde{x}}\right.\\ &\left.+\frac{1}{\tilde{R}}\frac{\partial\tilde{v}}{\partial\tilde{y}}+\frac{1}{\left(1+\frac{\tilde{y}}{\tilde{R}}\right)}\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2}-\frac{\tilde{v}}{\tilde{R}^2\left(1+\frac{\tilde{y}}{\tilde{R}}\right)}+\frac{1}{\tilde{R}^2\left(1+\frac{\tilde{y}}{\tilde{R}}\right)^2}\frac{d\tilde{R}}{d\tilde{x}}\left[\tilde{u}+\frac{\tilde{y}\partial\tilde{v}}{\partial\tilde{x}}\right]\right\}\end{aligned}\right\} \quad (2)$$

$$\frac{\partial\tilde{u}}{\partial\tilde{x}}+\frac{\partial}{\partial\tilde{y}}\left[\left(1+\frac{\tilde{y}}{\tilde{R}}\right)\tilde{v}\right]=0, \quad (3)$$

where $\tilde{x}$ and $\tilde{y}$ are the co-ordinates parallel and normal to the surface respectively, $\tilde{u}$ is the velocity along the $\tilde{x}$-axis while $\tilde{v}$ is perpendicular to it and $\tilde{R}(\tilde{x})$ represents the local radius of curvature of the wall. When $\tilde{R} > 0$ the wall is convex outwards and when $\tilde{R} < 0$ the wall is concave.

If one assumes a characteristic jet width, δ, small in comparison with $\tilde{R}$ and no large variations in curvature occur so that $d\tilde{R}/d\tilde{x}$ is of order unity, the above equations may be simplified by retaining terms to order δ in the first and third equation and to order unity in the second equation. Neglecting higher order terms in the second equation is consistent with the rest of the analysis since the latter equation is integrated across the jet to obtain $\partial\tilde{p}/\partial\tilde{x}$; thus it is further reduced by order δ.

After defining dimensionless variables

$$u=\frac{\tilde{u}}{U},\quad v=\frac{\tilde{v}}{U},\quad x=\frac{\tilde{x}U}{\nu},\quad y=\frac{\tilde{y}U}{\nu},\quad R=\frac{\tilde{R}U}{\nu},\quad p=\frac{\tilde{p}}{\rho U^2}$$

one may write equations (1) to (3) as

$$u\frac{\partial u}{\partial x}+\left(1+\frac{y}{R}\right)v\frac{\partial u}{\partial y}+\frac{uv}{R}=-\frac{\partial p}{\partial x}+\left(1+\frac{y}{R}\right)\frac{\partial^2 u}{\partial y^2}+\frac{1}{R}\frac{\partial u}{\partial y}, \quad (4)$$

$$\frac{u^2}{R}=\frac{\partial p}{\partial y}, \quad (5)$$

$$\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\left[\left(1+\frac{y}{R}\right)v\right]=0, \quad (6)$$

where U is a constant reference velocity.

The boundary conditions representative of the flow of a jet over a surface are

$$\left.\begin{aligned} y = 0, &\quad u = v = 0; \\ y \to \infty, &\quad u = 0. \end{aligned}\right\} \tag{7}$$

In search for similarity a stream function may be defined by

$$u = \frac{\partial \psi}{\partial y}, \quad \left(1+\frac{y}{R}\right) v = -\frac{\partial \psi}{\partial x},$$

where $\psi = x^m f(\eta)$; $\eta = cy/x^n$ and the radius of curvature $R = \alpha x^n$.

Equation (4) becomes

$$x^{2m-2n-1}\left[c(m-n)(f')^2 - cmff'' + \frac{1}{(\alpha+\eta/c)}(n\eta(f')^2 - mff') - \frac{1}{\alpha}\int_\eta^\infty \left[(2m-3n)(f')^2 - 2n\eta f'f''\right] d\eta\right] - cx^{m-3n}\left[c\left(1+\frac{\eta}{\alpha c}\right)f''' + \frac{1}{\alpha}f''\right] = 0, \tag{8}$$

where (5) was used to express the pressure term as a function of the velocity field and the radius of curvature. For similar solutions to exist the equation should become independent of x, which leads to the requirement

$$m + n = 1. \tag{9}$$

Thus one relation between the similarity exponents has been determined and one more relation is required.

A second relation cannot be obtained by any simple principle such as the constancy of momentum flux. For the case of a wall jet over an infinite flat plate, Glauert (1956) derived an integral invariant by considering the exterior momentum flux and this invariant gave a second relation between the similarity exponents. An analogous procedure is adopted here.

Consider the integral of (4) with respect to y between the limits of y and ∞ and use the condition that $u \to 0$ as $y \to \infty$. Then, using the continuity equation, one obtains

$$\frac{\partial}{\partial x}\int_y^\infty u^2 dy - \left(1+\frac{y}{R}\right)uv + \frac{1}{R}\int_y^\infty uv\,dy = \int_y^\infty \frac{\partial}{\partial x}\left\{\frac{1}{R}\int_y^\infty u^2 dy\right\}dy - \left(1+\frac{y}{R}\right)\frac{\partial u}{\partial y}. \tag{10}$$

Multiplying by u, integrating with respect to y from 0 to ∞, and simplifying gives

$$\frac{\partial}{\partial x}\int_0^\infty u\left[\left(1+\frac{y}{R}\right)\left\{\int_y^\infty u^2 dy\right\} - \int_y^\infty \frac{y}{R}u^2 dy\right]dy - \int_0^\infty \left(1+\frac{y}{R}\right)\frac{v}{R}\left\{\int_y^\infty u^2 dy\right\}dy + \frac{1}{R}\int_0^\infty u\left\{\int_y^\infty uv\,dy\right\}dy - \frac{1}{2R}\int_0^\infty u^2 dy = 0. \tag{11}$$

An invariant is obtained if

$$\int_0^\infty \left(1+\frac{y}{R}\right)v\left\{\int_y^\infty u^2 dy\right\}dy + \frac{1}{2}\int_0^\infty u^2 dy - \int_0^\infty u\left\{\int_y^\infty uv\,dy\right\}dy = 0, \tag{12}$$

or, in terms of the similarity variables, if

$$\int_0^\infty \frac{ff'(1-n)f - n\eta f'}{\left(1+\dfrac{\eta}{\alpha c}\right)}d\eta + \tfrac{1}{2}c\int_0^\infty (f')^2 d\eta + n\int_0^\infty \eta f(f')^2 d\eta - \int_0^\infty f\left\{\int_\eta^\infty (f')^2 d\eta\right\}d\eta = 0. \tag{13}$$

Provided (13) is satisfied, then

$$\frac{\partial}{\partial x}\int_0^\infty u\left\{\left(1+\frac{y}{R}\right)\int_y^\infty u^2dy-\frac{1}{R}\int_y^\infty yu^2dy\right\}dy = 0. \tag{14}$$

In the case of a similarity solution, (14) shows that

$$n = 3m, \tag{15}$$

and a second relation between the similarity exponents has been obtained. Thus $m = \frac{1}{4}$ and $n = \frac{3}{4}$. Equation (8) may now be rewritten

$$f'''+ff''+2(f')^2 + \frac{1}{\alpha}\left\{4\eta f'''+4f''+\frac{4f'}{1+(4\eta/\alpha)}\left[f-3\eta f'\right]-16\int_\eta^\infty (f')^2d\eta+12\eta(f')^2\right\} = 0, \tag{16}$$

where the value of c was chosen as $\frac{1}{4}$ in order that the equation will reduce to that of Glauert as $\alpha\to\infty$.

The validity of (13) may be established as follows. Integrate (16) with respect to η between the limits 0 and ∞ to obtain

$$\frac{16}{\alpha}\int_0^\infty f\left\{\int_\eta^\infty (f')^2d\eta\right\}d\eta = \int_0^\infty f\left\{f'''+ff''+2(f')^2+\frac{1}{\alpha}\left[4\eta f'''+4f'' + \frac{4f'}{1+(4\eta/\alpha)}(f-3\eta f')+12\eta(f')^2\right]\right\}d\eta. \tag{17}$$

If one multiplies (13) by $16/\alpha$, uses the derived values of c and n, and substitutes for the double integral term from (17), the left-hand side of (13) takes the form

$$\int_0^\infty f\left\{f'''\left(1+\frac{4\eta}{\alpha}\right)+f''\left(f+\frac{4}{\alpha}\right)+2(f')^2\left(1-\frac{1}{\alpha f}\right)\right\}d\eta,$$

which when integrated by parts using the boundary conditions can be shown to be identically zero. Therefore, (16) satisfies the restriction and an invariant is obtained from (13) in the following dimensional form:

$$F = \nu^2U\int_0^\infty f'\left[\left(1+\frac{4\eta}{\alpha}\right)\int_\eta^\infty \tfrac{1}{4}(f')^2d\eta-\frac{1}{\alpha}\int_\eta^\infty \eta(f')^2d\eta\right]d\eta. \tag{18}$$

The constancy of F is strictly dependent on the velocity profiles being similar, whereas the existence of the corresponding invariant for flow over a flat surface (Glauert 1956) is not dependent on similarity. Although the similarity solution is self-consistent, the conditions under which F is invariant appear to be far more restrictive than those required for invariants in classical similarity solutions like the plane wall-jet or a free jet.

The integral term in (16) was removed by differentiation to yield

$$f^{\text{iv}}\left(1+\frac{4\eta}{\alpha}\right)+f'''\left(f+\frac{8}{\alpha}\right)+f'f''\left(5+\frac{24}{\alpha}\eta\right)+\frac{28}{\alpha}(f')^2 + \frac{4/\alpha}{(1+(4/\alpha)\eta)}\left[(f''f-6\eta f'f''-2(f')^2)-\frac{4/\alpha}{(1+(4/\alpha)\eta)}(ff'-3\eta(f')^2)\right] = 0, \tag{19}$$

with the boundary condition $f(0) = f'(0) = f'(\infty) = 0$ and a compatibility condition at the wall

$$f'''(0) = \frac{4}{\alpha}\left[-f''(0)+4\int_0^\infty (f')^2d\eta\right]. \tag{20}$$

If $f(\eta)$ is a solution satisfying the boundary conditions, so is $g(\eta) = Af(A\eta)$ for an arbitrary constant A. However, since the reference velocity, U, is undetermined, the solution may be normalized such that $f(\infty) = 1$, without a loss of generality.

Results and discussion

Equation (19) was solved numerically using a variable step Runge–Kutta method. The variation of f' with η is illustrated in figure 1 with $4/\alpha$ as the para-

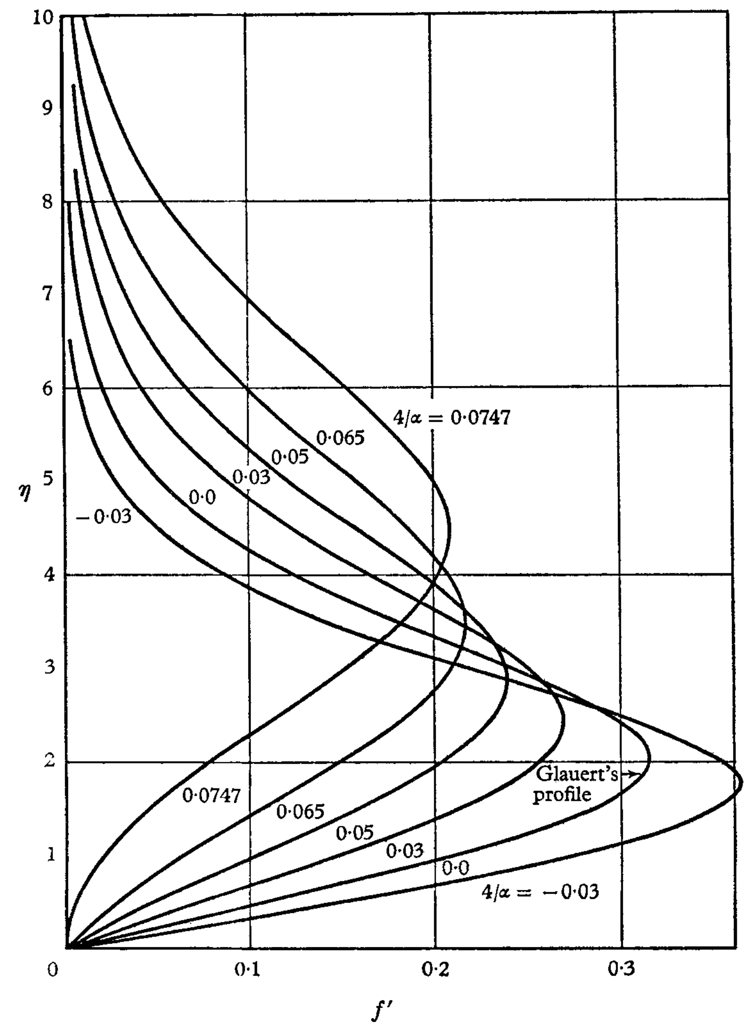

FIGURE 1. Variation of the velocity profile with pressure gradient.

meter of each curve. When $4/\alpha > 0$, the surface is convex and the jet is subjected to an adverse pressure gradient. Conversely, when $4/\alpha < 0$, the surface is concave and the pressure gradient is favourable. Glauert's (1956) solution of the wall jet over a plane surface is obtained as a special case when $4/\alpha = 0$. As $4/\alpha$ increases the jet becomes wider, the velocity profile develops an inflexion point near the wall and the skin friction, which is represented by $f''(0)$, is reduced (figure 2). The separation profile is obtained when $4/\alpha \simeq 0{\cdot}075$. The value of η,

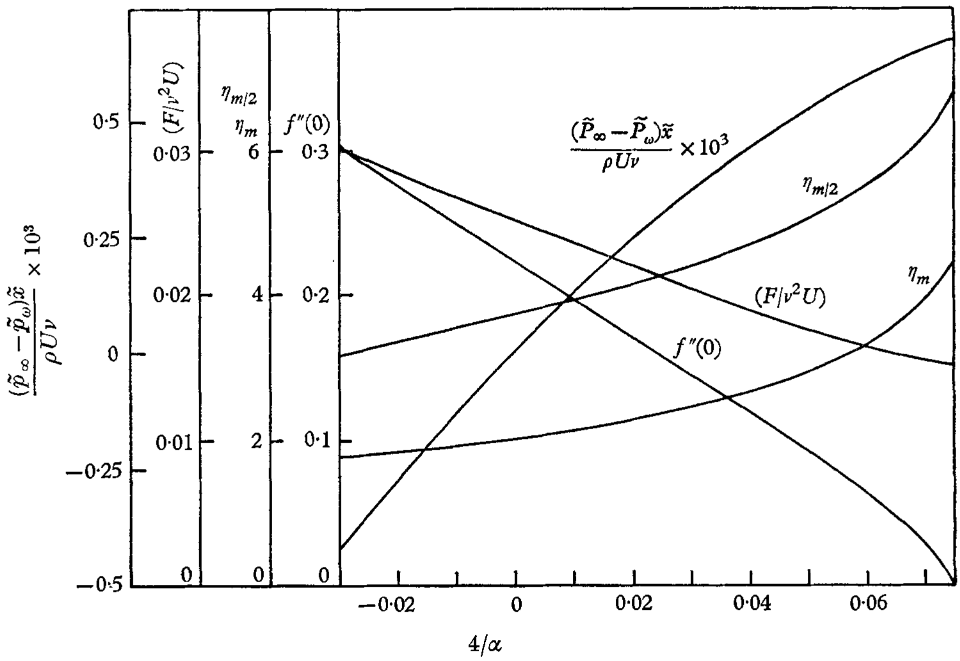

FIGURE 2. Variation of some properties of the flow with pressure gradient.

η_m, corresponding to the maximum value of f' and $\eta_{m/2}$, corresponding to one-half of the maximum value of f' on the far side from the wall represent characteristic widths of the velocity profile and are plotted in figure 2. The pressure difference across the jet, which in this case is inversely proportional to x, is also plotted on the figure.

The arbitrary reference velocity may be determined by using the invariant F from (18). The value of $F/\nu^2 U$ as a function of the curvature parameter, $4/\alpha$, was determined by evaluating the integral numerically and the results are plotted in figure 2. Should the similarity solution be verified by an appropriate experiment, the value of F could be determined directly from the measured velocity profiles after the wall jet has achieved a similar state. The reference velocity would then be determined and the results of the analysis presented here could be used to predict the behaviour of the downstream development of the flow.

The validity of the order of magnitude approximations used in simplifying the equations of motion will now be considered. If the characteristic jet width is defined as the distance from the wall to the location of the velocity maximum, then $\delta = ym/R = (4/\alpha)\eta_m$. The largest value of δ is $0{\cdot}328$ and this occurs at separation. Consequently, neglecting terms of order δ^2 *may* lead to an error of about 10%. One may also consider terms like $(uv/R)/(\nu\, \partial u/\partial y)$ which, according to the order of magnitude approximations, should be of order δ. From the similarity solution it can be shown

$$\left(\frac{uv}{R}\right)\bigg/\left(v\frac{\partial u}{\partial y}\right) = \frac{4}{\alpha}\frac{f'}{f''} \approx \frac{4}{\alpha},$$

except near the velocity maximum where $f''(\eta_m) = 0$. At separation, $4/\alpha \approx 0{\cdot}075$, and if $\delta = O(|4/\alpha|)$, then the terms dropped may be of order of 1 %.

The present analysis may serve as a basis for an approximate method of solution for a wall jet over an arbitrary surface in much the same way as the Falkner–Skan solution does in the boundary layer case.

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